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Estimability measures and their application to GPS/INS

Jeong Hyun Moon[1], Sinpyo Hong[2,*], Ho-Hwan Chun[3] and Man Hyung Lee[4]

[1]*Department of Mechanical and Intelligent Systems Engineering, Pusan National University*
[2]*General Marine Business Inc, Ulsan 680-819, Korea*
[3]*Department of Naval Architecture and Ocean Engineering, Pusan National University*
[4]*School of Mechanical Engineering, Pusan National University, Busan 609-735, Korea*

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Abstract

In this paper, estimability measures and their properties are introduced for a discrete linear system. The degrees of estimability of both a system and its subspaces can be examined with these measures for multi-input/multi-output time-varying systems. The relations among estimability, error covariance, and information matrix are presented. The effects of the initial error covariance, system model perturbation, and the transformation of state variable on the measures are also given. The estimability measures are applied to GPS/INS to confirm the estimability analysis results.

Keywords: Estimability measure; Observability measure; Singular value decomposition (SVD); The global positioning system (GPS); Inertial navigation system (INS)

1. Introduction

In estimation applications, the performance of estimators is usually examined with covariance analysis. Error covariance provides statistical information on the behavior of state estimators [1]. To characterize the properties of error covariance, the concept of estimability was introduced by Baram and Kailath [2]. For systems free from plant noise, Goshen-Meskin and Bar-Itzhack [3] showed that the conditions for estimability and observability are the same. Error covariance can be used to test the degree of observability [4]. Especially, covariance analysis has widely been applied in the observability study of the inertial navigation systems (INS) [5-10].

Even though estimability has close relationships with observability, it has its own characteristics [11, 12]. Both the decrease and decrease rate of error covariance can be greatly influenced by the choice of the initial error covariance. The direction of unestimable subspace can be different from that of unobservable subspace. In this paper, properties of error covariance are investigated with estimability measures. The degree of estimability and the effects of both system model perturbation and transformation of state variable on error covariance are studied with the measures.

Two types of estimability measures are considered for a discrete linear system. The measures are defined as the ratio of error covariance change to the initial error covariance. One measure is concerned with the estimability of a subspace in the state-space. This measure describes the relative error covariance change in a specific direction. The other measure is defined with the minimum of the subspace estimability measure over the whole state-space. Thus, it is the estimability measure of a system. These measures are applicable to multi-input/multi-output time-varying systems.

A physical system and its mathematical model are usually not the same. Uncertainties in the systems

*Corresponding author. Tel.: +82 52 270 3559, Fax.: +82 52 256 8053
E-mail address: sinpyo59@paran.com
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may cause modeling errors. For complicated systems, simple models are often preferred to make analysis of the system uncomplicated. For real-time estimation, simple models are also convenient in order to save computational loads. In these cases, system designers may be interested in the influence of system model perturbation on the estimation. The influence of model perturbation on the estimability measures is inspected with the singular value decomposition (SVD) techniques in this paper.

There can be several mathematical models for a given physical system. The choice of the mathematical model may depend on the application area of the model. For INS, there are several linearized models for navigation error propagation [13-17]. The selection of the INS error propagation model depends on both the reference frames of the navigation mechanization equations and attitude error model. Thus, navigation system designers might be often interested in the influence of the navigation error propagation models on the error covariance. For the error models that describe the same navigation error states with different coordinate systems, there exists a one-to-one linear transformation between each pair of the error propagation models. In this paper, the effect of constant linear transformations of the navigation error state on the error covariance is studied with the estimability measures.

The proposed estimability measures are applied to GPS/INS. Numerical tests are carried out to verify the estimability analysis results for a constant speed horizontal motion. The effects of model perturbation, reference coordinate system, and initial error covariance on the error covariance are examined with the measures.

2. Estimability measures

In this section, estimability measures and their properties are introduced for a discrete linear system. The relations among estimability, error covariance, and information matrix are examined with the SVD. Both the effects of the perturbation of the system model and the transformation of the state vector on measures are also presented.

Before the main results are given, notations for estimation applications and matrix analysis are introduced. $\hat{x}$ denotes the estimate of a random variable x; $\tilde{x}$ denotes the estimation error, $\hat{x}-x$; both $E[x]$ and $\bar{x}$ indicate the expected value or mean of a random variable x; $x \sim N(\bar{x}, P)$ denotes x is a normal random variable with the mean $\bar{x}$ and covariance P. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the vector space of real n-vectors, $\mathbb{R}^{m\times n}$ denotes the set of $m\times n$ real numbers, $(\bullet)^T$ denotes the transpose of a matrix or vector, $(\bullet)^{-1}$ denotes the inverse of a square matrix. Note that for a matrix M, $(M^T)^{-1}=(M^{-1})^T$. $\mathrm{diag}(d_1,\cdots,d_n)$ denotes a matrix with zero off-diagonal elements; diagonal elements are d_1 , $\cdots$, d_n from left to right. I_n denotes $n\times n$ identity matrix, 0 denotes a zero matrix with an appropriate dimension, $\|\bullet\|_2$ denotes the 2-norm of a matrix or vector.

For a matrix $M\in\mathbb{R}^{m\times n}$, the SVD, $M=UDV^T$, exists such that $U\in\mathbb{R}^{m\times m}$ and $V\in\mathbb{R}^{n\times n}$ are orthogonal, $D=\mathrm{diag}(\sigma_1,\sigma_2,\cdots,\sigma_p)\in\mathbb{R}^{m\times n}$ with $p=\min\{m,n\}$ and $\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_p\ge 0$. The σ_i are the singular values of M. Columns u_i in U are called the left singular vectors of M and columns v_i in V are called the right singular vectors of M. It can be shown that $M=\sum_{i=1}^{p}\sigma_i u_i v_i^T$. If M is symmetric and non-negative, then $U=V$ and the left and right singular vectors are the same and simply called singular vectors. $\bar{\sigma}(\bullet)$ denotes the largest singular value of a matrix. Note that $\|\bullet\|_2=\bar{\sigma}(\bullet)$. $\underline{\sigma}(\bullet)$ denotes the smallest singular value of a matrix.

Consider the following system:

$$\left.\begin{aligned} x_i &= \Phi_{i,0}x_0 \\ y_i &= H_i x_i + v_i \end{aligned}\right\} \tag{1}$$

where $x_i\in\mathbb{R}^n$ is the state vector at the time step i, $x_0\in\mathbb{R}^n$ is the initial state vector, $\Phi_{i,0}\in\mathbb{R}^{n\times n}$ is the state transition matrix from the time step 0 to the time step i, $y_i\in\mathbb{R}^m$ is the measurement vector at the time step i, $v_i\in\mathbb{R}^m$ is the measurement noise vector at the time step i, and $H_i\in\mathbb{R}^{m\times n}$ is the measurement matrix at the time step i. Assume that $x_0\sim N(\bar{x}_0,P_0)$ with $P_0>0$, $v_i\sim N(0,R_i)$ with $R_i>0$, $E\left[v_i v_j^T\right]=0$ for $i\ne j$, and $E\left[v_i x_0^T\right]=0$. The conditional expected-value estimate that is also the minimum variance or maximum-likelihood estimate of the above system with a set of measurements $\{y_0,y_1,\cdots,y_k\}$, $\hat{x}_{0,k}$, is given as [18, 19]

$$\hat{x}_{0,k}=\left(P_0^{-1}+L_{0,k}\right)^{-1}\left(K_k+P_0^{-1}\bar{x}_0\right) \tag{2}$$

with

$$L_{0,k}=\sum_{i=0}^{k}\Phi_{i,0}^T H_i^T R_i^{-1} H_i \Phi_{i,0},\; K_k=\sum_{i=0}^{k}\Phi_{i,0}^T H_i^T R_i^{-1} y_i \tag{3}$$

where $L_{0,k}$ is the observability gramian or information matrix. Note that if the measurements do not have noise, then the system is deterministic and the corresponding observability gramian is

$$\mathcal{L}_{0,k} = \sum_{i=0}^{k} \Phi_{i,0}^T H_i^T H_i \Phi_{i,0} \tag{4}$$

The above system is observable on $[0,k]$ if and only if $L_{0,k} > 0$ [20]. If the system is unobservable, then the null space of $L_{0,k}$ is called the unobservable subspace of the system. Let $\tilde{x}_{0,k} = x_0 - \hat{x}_{0,k}$. The error covariance matrix is defined as

$$P_{0,k} \triangleq E\left[\tilde{x}_{0,k}\tilde{x}_{0,k}^T\right] \tag{5}$$

Then, we have

$$\left(P_{0,k}\right)^{-1} = P_0^{-1} + L_{0,k} \tag{6}$$

Hence,

$$\left(P_0 - P_{0,k}\right)\left(P_0\right)^{-1} = P_{0,k}L_{0,k} \tag{7}$$

In estimation applications, the behavior of the error covariance is one of the main concerns of estimator designers. To characterize estimator performance, estimability and its measures were introduced in [12]. The following is a brief summary of the estimability properties given in [12]. A system is called estimable if $P_0 - P_{0,k} > 0$. The null space of $P_0 - P_{0,k}$ is an unestimable subspace. A system is observable if and only if the system is estimable. If the span of a vector u is unobservable, then the span of $P_0^{-1}u$ is unestimable. Thus, the unestimable subspace may not be the same as the unobservable subspace for a given system. A measure of estimability for a subspace is defined with the ratio of the decrease in the error covariance of a vector in the subspace to the initial error covariance of the vector:

$$\nu\left(L_{0,k}, P_0, u\right) = \frac{u^T\left(P_0 - P_{0,k}\right)u}{u^T P_0 u} \tag{8}$$

where $u \in \mathbb{R}^n$. Thus, the measure indicates relative decrease of error covariance in the direction of the subspace. Then, the measure of estimability for the system can be defined with

$$\underline{\nu}\left(L_{0,k}, P_0\right) \triangleq \min_{u\in\mathbb{R}^n} \nu\left(L_{0,k}, P_0, u\right) \tag{9}$$

Therefore, the system is unestimable if and only if $\underline{\nu}\left(L_{0,k}, P_0\right) = 0$. The concept of the estimability is similar to that in [2]. However, estimability in [2] is concerned with the error covariance rather than the initial error covariance.

In the following, new properties of estimability are examined in detail. The measures of estimability are closely related to the SVD of $\sqrt{P_0}L_{0,k}\sqrt{P_0}$. Let the SVD of $\sqrt{P_0}L_{0,k}\sqrt{P_0}$ be $U_p\Sigma_p U_p^T$ where Σ_p is $\text{diag}\left(\sigma_{p,1}, \sigma_{p,2}, \cdots, \sigma_{p,n}\right) \in \mathbb{R}^{n\times n}$ and $U_p = \left[u_{p,1}u_{p,2}\cdots u_{p,n}\right]$. Then $P_{0,k} = \sqrt{P_0}U_p\left[I_n + \Sigma_p\right]^{-1}U_p^T\sqrt{P_0}$. Let $d_{p,i} = \sigma_{p,i}/\left(1+\sigma_{p,i}\right)$. Then, $d_{p,1} \ge d_{p,2} \ge \cdots \ge d_{p,n} \ge 0$. Let $D_p = \text{diag}\left(d_{p,1}, d_{p,2}, \cdots, d_{p,n}\right) \in \mathbb{R}^{n\times n}$. Then, $P_0 - P_{0,k} = \sqrt{P_0}U_p D_p U_p^T\sqrt{P_0}$. Thus,

$$\nu\left(L_{0,k}, P_0, u\right) = \frac{u^T\sqrt{P_0}U_p D_p U_p^T\sqrt{P_0}u}{u^T P_0 u} \tag{10}$$

Let $z = \sqrt{P_0}u/\left\|\sqrt{P_0}u\right\|_2$. Then, $\|z\|_2 = 1$ and

$$\nu\left(L_{0,k}, P_0, u\right) = z^T U_p D_p U_p^T z \tag{11}$$

Thus, we have

$$\nu\left(L_{0,k}, P_0, \sqrt{P_0^{-1}}u_{p,i}\right) = d_{p,i} \tag{12}$$

$$d_{p,n} \le \nu\left(L_{0,k}, P_0, u\right) \le d_{p,1} \tag{13}$$

When $u = \sqrt{P_0^{-1}}u_{p,n}$, $\nu\left(L_{0,k}, P_0, u\right)$ takes its minimum value such that

$$\underline{\nu}\left(L_{0,k}, P_0\right) = d_{p,n} \tag{14}$$

Therefore, we have the following theorem:

Theorem 1:

$$\underline{\nu}\left(L_{0,k}, P_0\right) = \frac{\underline{\sigma}\left(\sqrt{P_0}L_{0,k}\sqrt{P_0}\right)}{1 + \underline{\sigma}\left(\sqrt{P_0}L_{0,k}\sqrt{P_0}\right)} \tag{15}$$

From Eq. (13) we have the following theorem:

Theorem 2: Let $u \in \mathbb{R}^n$. Then,

$$\frac{\underline{\sigma}\left(P_0\right)\underline{\sigma}\left(L_{0,k}\right)}{1 + \underline{\sigma}\left(L_{0,k}\right)\overline{\sigma}\left(P_0\right)} \le \nu\left(L_{0,k}, P_0, u\right) \le \frac{\overline{\sigma}\left(P_0\right)\overline{\sigma}\left(L_{0,k}\right)}{1 + \overline{\sigma}\left(L_{0,k}\right)\underline{\sigma}\left(P_0\right)}$$

Proof: Since

$$\sigma_i\left(L_{0,k}\right)\underline{\sigma}\left(P_0\right) \le \sigma_i\left(\sqrt{P_0}L_{0,k}\sqrt{P_0}\right) \le \sigma_i\left(L_{0,k}\right)\overline{\sigma}\left(P_0\right)$$

[21, 22], we have

$$\frac{\underline{\sigma}\left(P_0\right)\underline{\sigma}\left(L_{0,k}\right)}{1 + \underline{\sigma}\left(L_{0,k}\right)\overline{\sigma}\left(P_0\right)} \le d_{p,n} \quad \text{and}$$

$$d_{p,1} \le \frac{\overline{\sigma}\left(P_0\right)\overline{\sigma}\left(L_{0,k}\right)}{1 + \overline{\sigma}\left(L_{0,k}\right)\underline{\sigma}\left(P_0\right)} \qquad \blacksquare$$

Theorem 2 shows that the upper and lower bounds of the relative covariance decrease are proportional to the smallest and the largest singular values of the initial error covariance matrix, respectively. The following theorems show the sensitivity of the estimability measures to a perturbation in the information matrix:

Theorem 3:
Let $\Delta \in \mathbb{R}^{n\times n}$, $r \triangleq 1+\underline{\sigma}(L_{0,k})\underline{\sigma}(P_0) > \bar{\sigma}(\Delta)$, and $u \in \mathbb{R}^n$. Then,

$$\left|\nu(L_{0,k}+\Delta, P_0, u) - \nu(L_{0,k}, P_0, u)\right| \le \frac{\bar{\sigma}(\Delta)\bar{\sigma}(P_0)}{r(r-\bar{\sigma}(\Delta))} \quad (16)$$

Proof: Let $z = \sqrt{P_0}u \big/ \left\|\sqrt{P_0}u\right\|_2$. Then, $\|z\|_2 = 1$. Let $L_p = \sqrt{P_0}L_{0,k}\sqrt{P_0}$, $\Delta_p = \sqrt{P_0}\Delta\sqrt{P_0}$, and $M = (I_n + L_p)^{-1} - (I_n + L_p + \Delta_p)^{-1}$. Then, $\left|\nu(L_{0,k}+\Delta, P_0, u) - \nu(L_{0,k}, P_0, u)\right| = \left|z^T M z\right| \le \|M\|_2$. From Theorem 2.3.4 of [22], we have $\|M\|_2 \le \|\Delta_p\|_2 \left\|(I_n + L_p)^{-1}\right\|_2^2 \Big/ \left(1 - \left\|(I_n+L_p)^{-1}\Delta\right\|_2\right)$. The proof of the theorem follows from the relations $\|\Delta_p\|_2 \le \|P_0\|_2\|\Delta\|_2$, $\left\|(I_n+L_p)^{-1}\right\|_2 \le 1\big/\left(1+\underline{\sigma}(L_{0,k})\underline{\sigma}(P_0)\right) = 1/r$, and $\left\|(I_n+L_p)^{-1}\Delta\right\|_2 \le \|\Delta\|_2 \big/ \left(1+\underline{\sigma}(L_{0,k})\underline{\sigma}(P_0)\right) = \|\Delta\|_2/r < 1$. ■

Theorem 4:
$\Delta \in \mathbb{R}^{n\times n}$ is symmetric and $L_{0,k}+\Delta \ge 0$. Then,

$$\left|\underline{\nu}(L_{0,k}+\Delta, P_0) - \underline{\nu}(L_{0,k}, P_0)\right| \le \bar{\sigma}(\Delta)\bar{\sigma}(P_0) \quad (17)$$

Proof: From Eq. (15),

$$\underline{\nu}(L_{0,k}+\Delta, P_0) - \underline{\nu}(L_{0,k}, P_0) = \frac{\underline{\sigma}\left(\sqrt{P_0}(L_{0,k}+\Delta)\sqrt{P_0}\right) - \underline{\sigma}\left(\sqrt{P_0}L_{0,k}\sqrt{P_0}\right)}{1+\underline{\sigma}\left(\sqrt{P_0}L_{0,k}\sqrt{P_0}\right) + \underline{\sigma}\left(\sqrt{P_0}(L_{0,k}+\Delta)\sqrt{P_0}\right)\left(1+\underline{\sigma}\sqrt{P_0}L_{0,k}\sqrt{P_0}\right)} \quad (18)$$

Since the denominator of the right-hand side of (18) is greater than one, we have

$$\left|\underline{\nu}(L_{0,k}+\Delta, P_0) - \underline{\nu}(L_{0,k}, P_0)\right| \le \left|\underline{\sigma}\left(\sqrt{P_0}(L_{0,k}+\Delta)\sqrt{P_0}\right) - \underline{\sigma}\left(\sqrt{P_0}L_{0,k}\sqrt{P_0}\right)\right| \le \bar{\sigma}\left(\sqrt{P_0}\Delta\sqrt{P_0}\right)$$

The proof of the theorem follows from the relation

$$\bar{\sigma}\left(\sqrt{P_0}\Delta\sqrt{P_0}\right) \le \bar{\sigma}(P_0)\bar{\sigma}(\Delta)$$ ■

The theorems also show that the relative error covariance decrease can be sensitive to a perturbation in the information matrix if the initial error covariance has a large norm.

Several models are used to describe error propagation in the INS. For each pair of the models, an equivalent transformation can be obtained. In the following the effect of equivalent transformation on the estimability measures of linear systems is examined. Let $x_i' = Tx_i$ and $y_i' = Sy_i$ be the transformed state variable and measurement in the equivalent system model where $T \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{m\times m}$ are constant non-singular matrices. Then, from Eq. (1) we have a new equivalent system

$$\left.\begin{array}{l} x_i' = \Phi_{i,0}' x_0' \\ y_i' = H_i' x_i' + v_i' \end{array}\right\} \quad (19)$$

where $\Phi_{i,0}' = T\Phi_{i,0}T^{-1}$, $H_i' = SH_iT^{-1}$, $x_0' \sim N(\bar{x}_0', P_0') = N(T\bar{x}_0, TP_0T^T)$, $v_i' \sim N(0, R_i') = N(0, SR_iS^T)$. Then, the optimal estimate of x_0' with the set of measurements $\{y_0', y_1', \cdots, y_k'\}$, $\hat{x}_{0,k}'$, becomes

$$\hat{x}_{0,k}' = \left(P_0'^{-1} + L_{0,k}'\right)^{-1}\left(K_k' + P_0'^{-1}\bar{x}_0'\right) \quad (20)$$

with $P_0' = TP_0T^T$, $L_{0,k}' = \sum_{i=0}^{k}\Phi_{i,0}'^T H_i'^T R_i'^{-1} H_i' \Phi_{i,0}' = (T^{-1})^T L_{0,k}T^{-1}$, $K_k' = \sum_{i=0}^{k}\Phi_{i,0}'^T H_i'^T R_i'^{-1} y_i' = (T^{-1})^T K_k$. Thus, $\hat{x}_{0,k}' = T\hat{x}_{0,k}$. Let $\tilde{x}_{0,k}' = x_0' - \hat{x}_{0,k}'$. Then the corresponding error covariance matrix can be defined as

$$P_{0,k}' \triangleq E\left[(\tilde{x}_{0,k}')(\tilde{x}_{0,k}')^T\right] \quad (21)$$

Then,

$$P_{0,k}'^{-1} = P_0'^{-1} + L_{0,k}' = (T^T)^{-1}P_{0,k}^{-1}T^{-1} \quad (22)$$

Hence,

$$P_{0,k}' = TP_{0,k}T^T \quad (23)$$

The estimability measure of the transformed system for the subspace spanned by a vector $u' \in \mathbb{R}^n$ is

$$\nu'(L_{0,k}', P_0', u') = \frac{u'^T(P_0' - P_{0,k}')u'}{u'^T P_0' u'} = \frac{u'^T T(P_0 - P_{0,k})T^T u'}{u'^T TP_0T^T u'} \quad (24)$$

Thus, we have the following theorem.

Theorem 5: If $T \in \mathbb{R}^{n \times n}$ is a constant non-singular matrix and $u \in \mathbb{R}^n$, then

$$\nu'\left(L'_{0,k}, P'_0, \left(T^T\right)^{-1} u\right) = \nu\left(L_{0,k}, P_0, u\right) \quad (25)$$

Since $\nu'\left(L'_{0,k}, P'_0, \left(\sqrt{P_0} T^T\right)^{-1} u_{p,n}\right) = \underline{\nu}'\left(L'_{0,k}, P'_0\right) = d_{p,n} = \underline{\nu}\left(L_{0,k}, P_0\right)$, we have the theorem:

Theorem 6: If $T \in \mathbb{R}^{n \times n}$ is a constant non-singular matrix, then

$$\underline{\nu}'\left(L'_{0,k}, P'_0\right) = \underline{\nu}\left(L_{0,k}, P_0\right) \quad (26)$$

The above two theorems show that estimability measures are unchanged during an equivalent transformation. Many transformations in INS are combinations of symmetric scale transformations and orthogonal rotation transformations. For example, the transformation from the navigation frame to the earth-centered earth-fixed (ECEF) frame can be represented with $T = T_r T_s$ where T_r is a rotation transformation and T_s is a scale transformation. Thus, for this case $\left(T^T\right)^{-1}$ in Theorem 5 is such that $\left(T_s^T T_r^T\right)^{-1} = T_r T_s^{-1}$. Therefore, the order of transformations is unchanged and only scaling is inverted in the transformation $\left(T^T\right)^{-1}$.

3. An application to GPS/INS

In this section, an application of the estimability measures to GPS/INS is demonstrated. The influences of both the system model perturbation and the transformation of navigation error state on the measures for a simple vehicle motion are presented. The effect of the initial error covariance on the measures is also given.

Notations for INS in this section follow those in [13, 17]. For a vector P, P^a is the vector decomposed in a coordinate frame *a*. C_a^b denotes the rotation matrix from a frame *a* to a frame *b*. ω_{ab}^c denotes the column vector of the angular velocity of a frame *b* relative to a frame *a,* decomposed in a frame *c*. Ω_{ab}^c denotes the cross product matrix of ω_{ab}^c. *i*, *e, n,* and *b* used for coordinate frames denote the earth-centered inertial (ECI) frame, ECEF frame, body-fixed navigation frame (north, east, down), and body frame (forward, right, down), respectively. For simplicity of notation, exponentiation with 10 as a base is expressed with E such that 2.0E-05 means 2.0×10^{-5} in this section.

In the following estimability analysis of strap-down INS, the navigation system error is the difference between the computed estimate and true value. The relatively exact state space model for error dynamics in the strap-down INS mechanization equation in the ECEF frame can be [16, 17, 23]

$$\dot{x}_e = F_e x_e + w_e,\; y_e = H_e x_e + v_e \quad (27)$$

with

$$x_e = \left[\left(\delta P^e\right)^T \quad \left(\delta V^e\right)^T \quad \left(\gamma^b\right)^T \quad \varepsilon_g^T \quad \varepsilon_a^T \quad \delta l^T\right]^T \quad (28)$$

$$F_e = \begin{bmatrix} 0 & I_3 & 0 & 0 & 0 & 0 \\ G^e & -2\Omega_{ie}^e & -C_b^e \mathcal{F}^b & 0 & C_b^e & 0 \\ 0 & 0 & -\Omega_{ib}^b & I_3 & 0 & 0 \\ 0 & 0 & 0 & -I_3/\tau_g & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_3/\tau_a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (29)$$

$$H_e = \left[I_3 \quad 0 \quad -C_b^e L^b \quad 0 \quad 0 \quad C_b^e\right]. \quad (30)$$

where x_e is the error state; w_e is the plant noise; y_e is the measurement estimation error; v_e is the measurement noise; δP^e and δV^e are the position and velocity errors; γ^b is the attitude error such that $\hat{C}_b^e = C_b^e\left(I_3 + \left[\gamma^b \times\right]\right)$; ε_g and ε_a are bias errors for the gyro and accelerometer, respectively; δl is the error in the computed estimate of l^b, the lever arm vector between the inertial sensors and GPS antenna; $G^e = \partial g^e / \partial P^e$ where g^e is the gravity; $\mathcal{F}^b$ and L^b are the cross product matrices of the specific force f^b and l^b, respectively; τ_g and τ_a are the correlation times of noises in the gyro and accelerometer, respectively.

A linearized error dynamics model of strap-down INS in the navigation frame can be

$$\dot{x}_g = F_g x_g + w_g,\; y_g = H_g x_g + v_g \quad (31)$$

with

$$x_g = \left[\left(\delta P^n\right)^T \quad \left(\delta V^n\right)^T \quad \left(\gamma^b\right)^T \quad \varepsilon_g^T \quad \varepsilon_a^T \quad \delta l^T\right]^T \quad (32)$$

$$F_g = \begin{bmatrix} \mathcal{H}_p & \mathcal{H}_n & 0 & 0 & 0 & 0 \\ G_n + V^n \times \mathcal{H}_{wp} & V^n \times \mathcal{H}_{wv} - \Omega_{en}^n - 2\Omega_{ie}^n & -C_b^n \mathcal{F}^b & 0 & C_b^n & 0 \\ -C_n^b \mathcal{H}_{wp2} & -C_n^b \mathcal{H}_{wv} & -\Omega_{ib}^b & I_3 & 0 & 0 \\ 0 & 0 & 0 & -I_3/\tau_g & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_3/\tau_a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (33)$$

$$H_g = \left[I_3 + \mathcal{H}_l \quad 0 \quad -\mathcal{H}_n C_b^n L^b \quad 0 \quad 0 \quad \mathcal{H}_n C_b^n \right] \tag{34}$$

where

$$\delta P^n = \begin{bmatrix} \delta\lambda \\ \delta\Phi \\ \delta h \end{bmatrix}, \; \delta V^n = \begin{bmatrix} \delta v_N \\ \delta v_E \\ \delta v_D \end{bmatrix}, \; \mathcal{H}_n = \begin{bmatrix} \frac{1}{R_\lambda} & 0 & 0 \\ 0 & \frac{1}{R_\Phi \cos\lambda} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathcal{H}_p = \begin{bmatrix} 0 & 0 & \frac{-v_N}{R_\lambda^2} \\ \frac{v_E \sin\lambda}{R_\Phi \cos^2\lambda} & 0 & \frac{-v_E}{R_\Phi^2 \cos\lambda} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{H}_{wv} = \begin{bmatrix} 0 & \frac{1}{R_\Phi} & 0 \\ \frac{-1}{R_\lambda} & 0 & 0 \\ 0 & \frac{-\sin\lambda}{R_\Phi \cos\lambda} & 0 \end{bmatrix},$$

$$\mathcal{H}_{wp} = \begin{bmatrix} -2\omega_{ie}\sin\lambda & 0 & \frac{-v_E}{R_\Phi^2} \\ 0 & 0 & \frac{v_N}{R_\lambda^2} \\ -2\omega_{ie}\cos\lambda - \frac{v_E}{R_\Phi \cos^2\lambda} & 0 & \frac{v_E \sin\lambda}{R_\Phi^2 \cos\lambda} \end{bmatrix},$$

$$G_n = \frac{\partial g^n}{\partial P^n}$$

$$\mathcal{H}_{wp2} = \begin{bmatrix} -\omega_{ie}\sin\lambda & 0 & \frac{-v_E}{R_\Phi^2} \\ 0 & 0 & \frac{v_N}{R_\lambda^2} \\ -\omega_{ie}\cos\lambda + \frac{v_E}{R_\Phi \cos^2\lambda} & 0 & \frac{v_E \sin\lambda}{R_\Phi^2 \cos\lambda} \end{bmatrix},$$

$$\mathcal{H}_l = \begin{bmatrix} 0 & 0 & \frac{-l_N}{R_\lambda^2} \\ \frac{l_E \sin\lambda}{R_\Phi \cos^2\lambda} & 0 & \frac{-l_E}{R_\Phi^2 \cos\lambda} \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_b^n l^b = \begin{bmatrix} l_N \\ l_E \\ l_D \end{bmatrix},$$

δP^n and δV^n are position and velocity error, respectively; $\delta\lambda$ is the latitude error, $\delta\Phi$ is the longitude error; δh is the altitude error; λ is the latitude; Φ is the longitude; R_λ is the sum of the radius of curvature in a meridian and the altitude; R_Φ is the sum of the transverse radius curvature and the altitude; γ^b in Eq. (32) is the same as that in Eq. (28) such that $\hat{C}_b^n = C_b^n\left(I + \left[\gamma^b \times\right]\right)$. If we define the attitude error in the navigation frame such that $\hat{C}_b^n = \left(I_3 - \left[\gamma^n \times\right]\right) C_b^n$, γ^n is the equivalent tilt angle of Φ model in the strap-down INS [24]. If nonlinear error terms are neglected, we have $\gamma^b = -R_n^b \gamma^n$. v_N, v_E, and v_D are the earth-relative velocity in the north, east, and downward directions, respectively; ω_{ie} is the earth rotation rate.

For a simplified model of the system $\left(F_g, H_g\right)$, the following system will be considered to study the effect of a system model perturbation on the estimability measures:

$$\dot{x}_s = F_s x_s + w_s, \; y_s = H_s x_s + v_s \tag{35}$$

where x_s has the same form of x_g and

$$F_s = \begin{bmatrix} 0 & \mathcal{H}_n & 0 & 0 & 0 & 0 \\ 0 & -2\Omega_{ie}^n & -C_b^n \mathcal{F}^b & 0 & C_b^n & 0 \\ 0 & 0 & -\Omega_{ib}^b & I_3 & 0 & 0 \\ 0 & 0 & 0 & -I_3/\tau_g & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_3/\tau_a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

$$H_s = \left[I_3 \quad 0 \quad -\mathcal{H}_n C_b^n L^b \quad 0 \quad 0 \quad \mathcal{H}_n C_b^n \right]$$

The linearized error propagation models $\left(F_e, H_e\right)$ and $\left(F_g, H_g\right)$ are not equivalent to each other in general. However, if relatively small terms in the error propagation equations such as $\mathcal{H}_p$, $\mathcal{H}_{wp}$, $\mathcal{H}_{wv}$, Ω_{en}^n, $\mathcal{H}_{wp2}$, and $\mathcal{H}_l$ are neglected, these error propagation models can be considered equivalent. The relations among x_e, x_g, y_e, and y_g can be stated as

$$x_e = T_{eg} x_g, \; y_e = S_{eg} y_g \tag{37}$$

where

$$T_{eg} = \begin{bmatrix} C_n^e \mathcal{H}_n^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & C_n^e & 0 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3 \end{bmatrix}, \; S_{eg} = C_n^e \mathcal{H}_n^{-1} \tag{38}$$

Thus, $x_g = T_{ge} x_e$ with $T_{ge} = T_{eg}^{-1}$. Note that if the vehicle velocity is not exceptionally high, both C_n^e and $\mathcal{H}_n$ can be considered time-invariant. Thus, both T_{ge} and T_{ge} can be considered time-invariant.

Comparisons are made on the estimability measures of the above three error dynamics models for a vehicle that moves at 100 m/s to the north during 17 seconds. Measurement sampling period is one second, and the standard deviation (STD) of GPS position measurement error vector in the navigation frame (North, East, Down) is (0.03, 0.03, 0.06) in meter, and the lever arm between the GPS antenna and inertial measurement unit (IMU) in the body frame is (1, 1, 1) in meter. τ_g and τ_a are 100 and 60 in seconds, respectively. Two initial error covariance matrices for the system (F_g, H_g), P_{0Tg} and P_{0Mg}, are considered for estimability study. P_{0Tg} corresponds to a tactical grade IMU and P_{0Mg} to a very low-grade micro electromechanical system (MEMS) IMU. Both of them are diagonal matrices with the following forms:

$$\begin{aligned} P_{0Tg} = \mathrm{diag}\Big(& \left(1/R_\lambda^2, 1/\left(R_\Phi \cos\lambda\right)^2, 1\right),\ 0.01^2(1,1,1), \\ & \left((5.0\text{E-}5)^2, (5.0\text{E-}5)^2, (0.5\pi/180)^2\right), \\ & \left(((1.0\text{E-}4)\pi/180\right)^2 (1,1,1), \\ & (5.0\text{E}-4)^2 (1,1,1),\ (1,1,1)\Big) \end{aligned} \tag{39}$$

$$\begin{aligned} P_{0Mg} = \mathrm{diag}\Big(& \left(1/R_\lambda^2, 1/\left(R_\Phi \cos\lambda\right)^2, 1\right),\ 0.1^2(1,1,1), \\ & \left((2.5\text{E-}3)^2, (2.5\text{E-}3)^2, (5\pi/180)^2\right), \\ & (0.1\pi/180)^2 (1,1,1), 0.025^2(1,1,1),\ (1,1,1)\Big) \end{aligned} \tag{40}$$

The corresponding initial error covariance matrices for the system (F_e, H_e), P_{0Te} and P_{0Me}, can be written as $P_{0Te} = T_{eg} P_{0Tg} T_{eg}^T$ and $P_{0Me} = T_{eg} P_{0Mg} T_{eg}^T$.

Estimability measures of the systems (F_e, H_e), (F_g, H_g), and (F_s, H_s) with the two initial error covariance matrices are shown in Tables 1 and 2. In the tables, L_e, L_g and L_s are the information matrices corresponding to the system (F_e, H_e), (F_g, H_g) and (F_s, H_s), respectively. $u_{sT,i}, u_{gT,i}$, $u_{eT,i}$, $u_{sM,i}$, $u_{gM,i}$, and $u_{eM,i}$ are the ith singular vectors of $\sqrt{P_{0Tg}} L_s \sqrt{P_{0Tg}}$, $\sqrt{P_{0Tg}} L_g \sqrt{P_{0Tg}}$, $\sqrt{P_{0Te}} L_e \sqrt{P_{0Te}}$, $\sqrt{P_{0Mg}} L_s \sqrt{P_{0Mg}}$, $\sqrt{P_{0Mg}} L_g \sqrt{P_{0Mg}}$, and $\sqrt{P_{0Me}} L_e \sqrt{P_{0Me}}$, respectively. e_i denotes the ith standard basis of a vector space such that $e_2 = [0\ 1\ 0 \cdots 0]^T$.

On the last seven columns of the first six rows in Table 2, the estimability measures are less than 0.01%. Thus, all of the three navigation error propagation models can be considered to have seven dimensional unestimable subspaces with both the tactical and very low-grade MEMS IMUs. The dimension of unestimable subspace is the same as that of unobservable subspace of a simplified error propagation model in ECEF frame in [17]. Estimability measures with the initial error covariance matrices corresponding to the tactical grade IMU for the vertical component of gyro bias error, $\nu\left(L_s, P_{0Tg}, e_{12}\right)$, $\nu\left(L_g, P_{0Tg}, e_{12}\right)$, and $\nu\left(L_e, P_{0Te}, T_{ge}^T e_{12}\right)$, are on the order of 1.0E-8. Thus, the gyro bias component is almost unestimable with the tactical grade IMU. However, for a very low-grade IMU, the estimability measures are in the order of 1.0E-4. Thus, the vertical component of gyro bias can be considered weakly estimable in this case. Estimability measures for yaw error, the 9th column of the last six rows in Table I, are in the order of 1.0E-3. Thus, yaw error is weakly estimable with both the tactical grade and low-grade MEMS IMUs. The tables show that estimability measures with the initial error covariance corresponding to the tactical grade IMU are smaller than those to the very low-grade MEMS IMU for standard bases except for position and lever arm errors. The tables also show that there are very small differences in the measures for L_s and L_g with the same initial error covariance and subspace directions. Thus, a small change in the navigation error model with the tactical and very low-grade MEMS IMUs causes small changes in estimability measures. Estimability measures with both the navigation and ECEF frames for equivalent standard bases are the same. Thus, the tables demonstrate that estimability measures are unchanged during the coordinate transformation between the navigation and ECEF frames.

4. Conclusion

In this paper, error covariance properties are investigated with estimability measures. Estimability analysis results show that the SVD of the product of the initial error covariance matrix and information matrix determines the characteristics of error covariance. It is proved that the estimability measures and their perturbation can be sensitive to the magnitude of the norm of the initial error covariance matrix. This indicates that estimability of a system can be influenced by the initial error covariance that is independent of system model. However, the measures are un-

Table 1. Estimability measures.

Index i	1	2	3	4	5	6	7	8	9
$\nu\left(L_s, P_{0Tg}, \sqrt{P_{0Tg}^{-1}}u_{sT,i}\right)$	1.00E+00	1.00E+00	1.00E+00	9.90E-01	9.90E-01	9.55E-01	5.90E-01	5.90E-01	1.66E-01
$\nu\left(L_g, P_{0Tg}, \sqrt{P_{0Tg}^{-1}}u_{gT,i}\right)$	1.00E+00	1.00E+00	1.00E+00	9.89E-01	9.89E-01	9.54E-01	5.64E-01	5.63E-01	1.35E-01
$\nu\left(L_e, P_{0Te}, \sqrt{P_{0Te}^{-1}}u_{eT,i}\right)$	1.00E+00	1.00E+00	1.00E+00	9.89E-01	9.89E-01	9.54E-01	5.64E-01	5.63E-01	1.35E-01
$\nu\left(L_s, P_{0Mg}, \sqrt{P_{0Mg}^{-1}}u_{sM,i}\right)$	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	9.91E-01
$\nu\left(L_g, P_{0Mg}, \sqrt{P_{0Mg}^{-1}}u_{gM,i}\right)$	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	9.90E-01
$\nu\left(L_e, P_{0Me}, \sqrt{P_{0Me}^{-1}}u_{eM,i}\right)$	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	1.00E+00	9.90E-01
$\nu\left(L_s, P_{0Tg}, e_i\right)$	5.00E-01	5.00E-01	5.00E-01	8.91E-01	8.92E-01	8.47E-01	3.24E-01	3.24E-01	2.52E-03
$\nu\left(L_g, P_{0Tg}, e_i\right)$	5.00E-01	5.00E-01	5.00E-01	8.91E-01	8.91E-01	8.57E-01	3.52E-01	3.51E-01	2.67E-03
$\nu\left(L_e, P_{0Te}, T_{ge}^T e_i\right)$	5.00E-01	5.00E-01	5.00E-01	8.91E-01	8.91E-01	8.57E-01	3.52E-01	3.51E-01	2.67E-03
$\nu\left(L_s, P_{0Mg}, e_i\right)$	4.98E-01	4.98E-01	5.00E-01	9.89E-01	9.90E-01	9.93E-01	4.86E-01	4.86E-01	8.03E-03
$\nu\left(L_g, P_{0Mg}, e_i\right)$	4.98E-01	4.98E-01	5.00E-01	9.89E-01	9.89E-01	9.92E-01	4.87E-01	4.87E-01	8.05E-03
$\nu\left(L_e, P_{0Me}, T_{ge}^T e_i\right)$	4.98E-01	4.98E-01	5.00E-01	9.89E-01	9.89E-01	9.92E-01	4.87E-01	4.87E-01	8.05E-03

Table 2. Estimability measures.

Index i	10	11	12	13	14	15	16	17	18
$\nu\left(L_s, P_{0Tg}, \sqrt{P_{0Tg}^{-1}}u_{sT,i}\right)$	2.28E-03	2.07E-03	1.82E-08	9.51E-12	1.78E-12	9.98E-16	6.01E-16	1.72E-15	2.02E-15
$\nu\left(L_g, P_{0Tg}, \sqrt{P_{0Tg}^{-1}}u_{gT,i}\right)$	2.38E-03	2.19E-03	3.44E-08	2.15E-08	3.36E-09	2.43E-09	1.37E-12	1.05E-12	6.05E-16
$\nu\left(L_e, P_{0Te}, \sqrt{P_{0Te}^{-1}}u_{eT,i}\right)$	2.38E-03	2.19E-03	3.31E-08	8.28E-09	2.43E-09	7.02E-17	1.29E-15	9.19E-13	5.57E-16
$\nu\left(L_s, P_{0Mg}, \sqrt{P_{0Mg}^{-1}}u_{sM,i}\right)$	9.82E-01	9.82E-01	8.51E-06	1.77E-08	2.85E-13	1.40E-12	1.10E-13	5.00E-14	1.08E-13
$\nu\left(L_g, P_{0Mg}, \sqrt{P_{0Mg}^{-1}}u_{gM,i}\right)$	9.81E-01	9.81E-01	3.38E-05	2.29E-05	9.12E-09	2.43E-09	1.30E-09	8.64E-13	1.21E-17
$\nu\left(L_e, P_{0Me}, \sqrt{P_{0Me}^{-1}}u_{eM,i}\right)$	9.81E-01	9.81E-01	3.38E-05	2.29E-05	2.43E-09	1.35E-09	1.56E-14	3.82E-14	2.71E-15
$\nu\left(L_s, P_{0Tg}, e_i\right)$	2.74E-02	2.73E-02	6.35E-08	3.37E-01	3.38E-01	2.74E-01	5.00E-01	5.00E-01	5.00E-01
$\nu\left(L_g, P_{0Tg}, e_i\right)$	2.68E-02	2.68E-02	6.36E-08	2.84E-01	2.84E-01	2.31E-01	5.00E-01	5.00E-01	5.00E-01
$\nu\left(L_e, P_{0Te}, T_{ge}^T e_i\right)$	2.68E-02	2.68E-02	6.36E-08	2.84E-01	2.84E-01	2.31E-01	5.00E-01	5.00E-01	5.00E-01
$\nu\left(L_s, P_{0Mg}, e_i\right)$	1.00E+00	1.00E+00	6.34E-04	5.06E-01	5.06E-01	9.98E-01	4.98E-01	4.98E-01	5.00E-01
$\nu\left(L_g, P_{0Mg}, e_i\right)$	9.99E-01	9.99E-01	6.35E-04	5.04E-01	5.05E-01	9.98E-01	4.98E-01	4.98E-01	5.00E-01
$\nu\left(L_e, P_{0Me}, T_{ge}^T e_i\right)$	9.99E-01	9.99E-01	6.35E-04	5.04E-01	5.05E-01	9.98E-01	4.98E-01	4.98E-01	5.00E-01

changed with a constant equivalent transformation of the system model.

An example on the application of estimability analysis to GPS/INS is given for a constant speed horizontal motion. It is confirmed that the estimability measures are less sensitive to system model perturbation with the tactical grade and very low-grade MEMS IMUs. It is also shown that the same measures are obtained for the navigation error propagation models with both the navigation reference frame and ECEF reference frame.

Estimability test results show that there is a seven-dimensional unestimable subspace with two initial error covariance matrices corresponding to the two IMUs. It is confirmed that error covariance changes are sensitive to the initial error covariance. With the initial error covariance for the tactical grade IMU, the vertical component of gyro bias can be considered unestimable. However, for the very low-grade MEMS IMU, it can be considered weakly estimable. In both cases, yaw error is weakly estimable.

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